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EIGENVALUES OF THE ANTIPLANE-SHEAR CRACK PROBLEM FOR A POWER-LAW MATERIAL

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This paper discusses the problem of finding the eigenvalue spectrum in determining the stress and strain fields at the tip of an antiplane-shear crack in a power-law material. It is shown that the perturbation method provides an analytical dependence of the eigenvalue on the material nonlinearity parameter and the eigenvalue of the linear problem. Thus, it is possible to find the entire spectrum of eigenvalues and not only the eigenvalue of the Hutchinson–Rice–Rosengren problem.

Key words: *antiplane-shear crack, power-law constitutive equations, eigenvalue, eigenvalue spectrum, perturbation method.*

1. Problem of Determining Eigenvalues in Nonlinear Fracture Mechanics. In modern nonlinear fracture mechanics, eigenvalue problems often arises in studies of crack-tip stress–strain fields in materials with nonlinear constitutive equations. For example, in studies of the stress and strain (strain rate) fields in a material with power-law constitutive relations (power-law nonlinear elastic strain, power-law plastic strain hardening law, the Bailey–Norton law of steady-state creep theory) using the eigenfunction expansion method, it is necessary to solve systems of nonlinear ordinary differential equations with the boundary conditions of no surface loads at the crack faces and symmetry conditions at the crack extension. It should be noted that the resulting system of nonlinear ordinary differential equations contains a parameter (eigenvalue) which needs to be determined in order to find a nontrivial solution of the system of ordinary differential equations that satisfies the indicated boundary conditions. In nonlinear fracture mechanics, one eigenvalue of the Hutchinson–Rice–Rosengren problem is known [1, 2]. In a material hardening according to the power law

$$\varepsilon_{ij} = (3/2)B\sigma_e^{n-1}s_{ij}, \tag{1.1}$$

where ε_{ij} are the strain tensor components, s_{ij} are the stress deviator components; σ_e is the stress intensity, B is a material constant determined experimentally, and n is the material nonlinearity parameter, the crack-tip stress field is represented as

$$\sigma_{ij}(r,\theta) = (J/(BI_nr))^{1/(n+1)}\tilde{\sigma}_{ij}(\theta,n). \tag{1.2}$$

Here J is the invariant integral of nonlinear fracture mechanics [3], I_n is the dimensionless J-integral, and $\tilde{\sigma}_{ij}(\theta,n)$ is the universal angular stress distribution determined by solving the boundary-value problem. Relations (1.2) represent the classical Hutchinson–Rice–Rosengren stress distribution at a crack tip in a material with the power-law constitutive relations (1.1).

The interest of researchers has, for a long time, focused on constructing higher-order approximations in asymptotic expansions of the stress and strain fields at a crack tip from the specified main term of the asymptotic expansion — the Hutchinson–Rice–Rosengren solution [4–10]. At present, however, it seems urgent to find the entire eigenvalue spectrum. Thus, in the problem of small-scale plastic flow with an asymptotic boundary condition at an infinitely distant point, it is necessary to study the asymptotic behavior of the far stress field more carefully

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and, hence, to determine the other eigenvalues in eigenfunction expansions of stresses (see [11]). Stepanova and Fedina [12] undertook an attempt to determine eigenvalues different from the eigenvalue of the Hutchinson–Rice–Rosengren problem for various values of the nonlinearity parameter. This was done using a numerical method for determining eigenvalues (the Runge–Kutta–Fel'berg method) in combination with a shooting method. In this case, the shooting is one-parametric and eigenvalues are easily determined.

Lu and Lee [13] studied the eigenvalue spectrum of the problem of extension of a space with a semi-infinite crack in a power-law material. As is shown in [13], it is insufficient to find one eigenvalue and higher-order approximations of the Hutchinson–Rice–Rosengren problem [4]. It should be noted that, from a mathematical point of view, the problems of opening mode and transverse shear cracks are more complex. In [13], eigenvalues are found numerically only for some exponents of the power law ($n = 3$, 5).

In the present work, it is shown analytical dependences of eigenvalues on the material nonlinearity parameter and the eigenvalue of the linear problem can be found using perturbation methods.

2. Formulation of the Problem. Basic Equations. We consider an antiplane-shear crack in a material with the power-law constitutive relations

$$\gamma_{rz} = (3/2)B\tau_e^{n-1}\tau_{rz}, \qquad \gamma_{\theta z} = (3/2)B\tau_e^{n-1}\tau_{\theta z}, \qquad \tau_e^2 = 3(\tau_{rz}^2 + \tau_{\theta z}^2), \tag{2.1}$$

where τ_e is the tangential stress intensity. In polar coordinates with pole at the crack tip, the equilibrium equation and the strain compatibility condition are written as

$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{\theta z}}{\partial \theta} = 0, \qquad \frac{\partial}{\partial r}(r\gamma_{\theta z}) = \frac{\partial \gamma_{rz}}{\partial \theta}. \tag{2.2}$$

Here and below, it is assumed that the stress and strain tensor components are normalized to τ_0 and γ_0, respectively (τ_0 is the limit of proportionality of the material if the constitutive relations considered describe nonlinear elastic or plastic strain in terms of plastic strain theory; γ_0 is the strain intensity corresponding to τ_0).

We introduce the stress function

$$\tau_{rz} = \frac{1}{r}\,\frac{\Phi(r,\theta)}{\partial\theta}, \qquad \tau_{\theta z} = -\frac{\partial\Phi(r,\theta)}{\partial r} \tag{2.3}$$

to satisfy the equilibrium equation identically. The strain compatibility condition implies a nonlinear partial differential equation for the function $\Phi(r,\theta)$. As is known, the use of power-law stress–strain relations leads to the separation of the variables r and θ; therefore, the solution of the problem is sought in the form

$$\Phi(r,\theta) = r^s f(\theta) + \ldots\ . \tag{2.4}$$

The stress tensor components become

$$\tau_{ij}(r,\theta) = r^{s-1}\tilde{\tau}_{ij}(\theta) + \ldots\ .$$

Substitution of (2.4) into (2.3), (2.1) and into the compatibility condition [the second equation of system (2.2)] yields the nonlinear ordinary differential equation for the function $f(\theta)$:

$$f''(nf'^2 + s^2 f^2) + f(C_1 f'^2 + C_2 f^2) = 0. \tag{2.5}$$

Here $C_1 = s(n-1)(2s-1) + s^2$ and $C_2 = s^3(n-1)(s-1) + s^4$.

Equation (2.5) with the boundary conditions

$$f\Big|_{\theta=\pm\pi} = 0 \tag{2.6}$$

defines the nonlinear eigenvalue problem: to find the value of the parameter s for which problem (2.5), (2.6) has a nontrivial solution.

3. Eigenvalues. The analytical expression for the eigenvalue s as a function of the material nonlinearity parameter n and the eigenvalue s_0 of the unperturbed linear problem ($n = 1$) can be found using the representation

$$s = s_0 + \varepsilon, \tag{3.1}$$

where ε is the deviation of the eigenvalue s from the eigenvalue s_0 under a change in n.

To estimate the effect of the change in n on the eigenvalue s, we represent the material nonlinearity parameter and the required function in the form

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \dots\ ; \tag{3.2}$$

$$f(\theta) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \dots\ , \tag{3.3}$$

where $f_0(\theta)$ is the solution of the linear problem ($n = 1$).

Substituting (3.1)–(3.3) into (2.5) and collecting the coefficients of the same powers of the small parameter ε, we obtain the following system of inhomogeneous linear ordinary differential equations for the functions $f_0(\theta)$, $f_1(\theta)$, $f_2(\theta)$, ... :

$$f_0'' + s_0^2 f_0 = 0; \tag{3.4}$$

$$f_1'' + s_0^2 f_1 = -s_0[n_1(s_0 - 1) + 2]f_0;$$

$$f_2'' + s_0^2 f_2 = -[(n_2 f_0'^2 + f_0^2)f_0'' + (C_1^2 f_0'^2 + C_2^2 f_0^2)f_0]/s_0^2 \quad \dots\ . \tag{3.5}$$

Here $C_1^2 = n_2 s_0^2 - 1 + 3n_1 s_0 + s_0 n_2(s_0 - 1)$ and $C_2^2 = s_0^3[n_2(s_0 - 1) + n_1]$.

The solution of Eq. (3.4) that satisfies the boundary conditions $f_0|_{\theta=\pm\pi} = 0$ has the form

$$f_0(\theta) = A\cos s_0\theta, \qquad s_0 = m/2, \qquad m = \pm 1, \pm 3, \pm 5, \dots \tag{3.6}$$

or

$$f_0(\theta) = B\sin s_0\theta, \qquad s_0 = m/2, \qquad m = 0, \pm 2, \pm 4, \dots\ .$$

Using the solution obtained for the function $f_0(\theta)$, one can successively find the functions $f_1(\theta)$, $f_2(\theta)$, ... Thus, we have the following boundary-value problem for determining the function $f_1(\theta)$:

$$f_1'' + s_0^2 f_1 = -s_0[n_1(s_0 - 1) + 2]f_0; \tag{3.7}$$

$$f_1\Big|_{\theta=\pm\pi} = 0. \tag{3.8}$$

Since the corresponding homogeneous problem has a nontrivial solution, to solve the inhomogeneous problem, we need to find a certain solvability condition which will allow us to determine the coefficients n_k of expansion (3.2).

4. Solvability Condition. It should be noted that in the use of perturbation methods, sets of problems arise which should be solved successively [14]. In this case, the first-order problem is usually homogeneous whereas higher-order problems are inhomogeneous but linear. If the corresponding homogeneous problem has a nontrivial solution, the inhomogeneous problem has no solution if the corresponding solvability condition is not satisfied.

Let us consider the boundary-value problem for the inhomogeneous linear ordinary differential equation of the second order

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x), \qquad a < x < b; \tag{4.1}$$

$$\alpha_{11}y'(a) + \alpha_{12}y(a) + \alpha_{13}y'(b) + \alpha_{14}y(b) = \beta_1, \quad \alpha_{21}y'(a) + \alpha_{22}y(a) + \alpha_{23}y'(b) + \alpha_{24}y(b) = \beta_2, \tag{4.2}$$

where the boundary operators are linearly independent, i.e., the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix}$$

has rank 2, and, hence, there is at least one nondegenerate matrix of size 2×2. Thus, at least one of the determinants

$$\Delta_{11} = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \qquad \Delta_{13} = \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{21} & \alpha_{23} \end{vmatrix}, \qquad \Delta_{14} = \begin{vmatrix} \alpha_{11} & \alpha_{14} \\ \alpha_{21} & \alpha_{24} \end{vmatrix},$$

$$\Delta_{23} = \begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{22} & \alpha_{23} \end{vmatrix}, \qquad \Delta_{24} = \begin{vmatrix} \alpha_{12} & \alpha_{14} \\ \alpha_{22} & \alpha_{24} \end{vmatrix}, \qquad \Delta_{34} = \begin{vmatrix} \alpha_{13} & \alpha_{14} \\ \alpha_{23} & \alpha_{24} \end{vmatrix}$$

is different from zero.

Generally, the boundary conditions may be mixed (or unseparated), i.e., they may contain the values of the required function and its derivative at the ends of the segment $[a, b]$. Taking into account the boundary conditions of the examined problem (3.8), we consider the case where the determinant $\Delta_{24} \neq 0$. Solution of Eqs. (4.2) for $y(a)$ and $y(b)$ yields

$$y(a) = \gamma_{11} y'(a) + \gamma_{12} y'(b) + \delta_1, \qquad y(b) = \gamma_{21} y'(a) + \gamma_{22} y'(b) + \delta_2, \tag{4.3}$$

where

$$\gamma_{11} = -\frac{\Delta_{14}}{\Delta_{24}}, \quad \gamma_{12} = -\frac{\Delta_{34}}{\Delta_{24}}, \quad \gamma_{21} = \frac{\Delta_{12}}{\Delta_{24}}, \quad \gamma_{22} = -\frac{\Delta_{23}}{\Delta_{24}},$$

$$\delta_1 = \frac{\beta_1 \alpha_{24} - \beta_2 \alpha_{14}}{\Delta_{24}}, \qquad \delta_2 = \frac{\beta_2 \alpha_{12} - \beta_1 \alpha_{22}}{\Delta_{24}}.$$

To find the solvability condition for the general case, we consider a conjugate problem. We multiply Eq. (4.1) by the function $u(x)$, which is called the conjugate solution to be determined. As a result, we obtain

$$p_2 u y'' + p_1 u y' + p_0 u y = g u, \qquad a < x < b.$$

Term-by-term integration of this relation from a to b yields

$$\int_a^b p_2 u y''\, dx + \int_a^b p_1 u y'\, dx + \int_a^b p_0 u y\, dx = \int_a^b g u\, dx.$$

Next, integrating the first two terms by parts and performing some transformations, we have

$$\int_a^b [p_2 u'' + (2p_2' - p_1) u' + (p_2'' - p_1' + p_0) u] y\, dx + \{p_2 u y' + [(p_1 - p_2') u - p_2 u'] y\}\Big|_a^b = \int_a^b g u\, dx. \tag{4.4}$$

Setting the integrand on the left of the last equality to zero, we obtain the following differential equation for the function u:

$$p_2 u'' + (2p_2' - p_1) u' + (p_2'' - p_1' + p_0) u = 0; \tag{4.5}$$

it is usually called conjugate with respect to the homogeneous equation (4.1). The homogeneous differential equation corresponding to (4.1) is called self-conjugate if it coincides with the equation conjugate to it (4.5). These equations coincide if the equalities $2p_2' - p_1 = p_1$, $p_2'' - p_1' = 0$ or $p_1 = p_2'$ are valid. In this case, the homogeneous equation corresponding to (4.1) has the form

$$p_2 y'' + p_2' y' + p_0 y = 0.$$

To determine the boundary conditions required to close the conjugate problem, we consider the homogeneous problem that corresponds to (4.1), (4.2). Then, relation (4.4) becomes (in the case of a self-conjugate equation)

$$\{p_2 [u y' - u' y]\}\Big|_{x=a}^{x=b} = 0.$$

Using equality (4.3), the last relation can be brought to the form

$$[-\gamma_{21} p_2(b) u'(b) - p_2(a) u(a) + \gamma_{11} p_2(a) u'(a)] y'(a)$$

$$+ [p_2(b) u(b) - \gamma_{22} p_2(b) u'(b) + \gamma_{12} p_2(a) u'(a)] y'(b) = 0.$$

We choose the boundary conditions of the conjugate problem in such a manner that each coefficient at $y'(a)$ and $y'(b)$ vanishes:

$$\begin{aligned} &p_2(a) u(a) - \gamma_{11} p_2(a) u'(a) + \gamma_{21} p_2(b) u'(b) = 0, \\ &p_2(b) u(b) + \gamma_{12} p_2(a) u'(a) - \gamma_{22} p_2(b) u'(b) = 0. \end{aligned} \tag{4.6}$$

Thus, the function u is a solution of the boundary-value problem for the equation

$$p_2u'' + p_2'u' + p_0u = 0 \tag{4.7}$$

with boundary conditions (4.6).

Having formulated the conjugate problem, we return to the initial inhomogeneous problem (4.1), (4.2) and find a solvability condition for it. Since the function u satisfies Eq. (4.7), expression (4.4) becomes

$$\{p_2[uy' - u'y]\}\Big|_{x=a}^{x=b} = \int_a^b gu\,dx.$$

Since the solution of the conjugate boundary-value problem u satisfies boundary conditions (4.2), the last relation can be written as

$$\delta_1 p_2(a)u'(a) - \delta_2 p_2(b)u'(b) = \int_a^b gu\,dx. \tag{4.8}$$

Equality (4.8) is the required solvability condition, where u is a solution of the conjugate boundary-value problem.

5. Solvability Condition of Problem (3.7), (3.8). Returning to the boundary-value problem for the inhomogeneous linear ordinary differential equation (3.7), whose solution should satisfy boundary conditions (3.8), it is easy to establish that this equation is self-conjugate since, in this case, $p_2 = 1$, $p_1 = 0$, and $p_2' = p_1$, and the solvability condition is formulated as follows:

$$\int_{-\pi}^{\pi} gu\,d\theta = 0. \tag{5.1}$$

Here g is the right side of Eq. (3.7):

$$g(\theta) = -s_0[n_1(s_0 - 1) + 2]f_0(\theta),$$

the function u is the conjugate solution coincident with the function $f_0(\theta)$. For odd numbers m, the function $f_0(\theta)$ is given by relation (3.6). In this case, the conjugate solution has the form

$$u = \cos s_0\theta. \tag{5.2}$$

It can be shown by simple calculations that the solvability condition is satisfied only by choosing the coefficient $n_1 = -2/(s_0 - 1)$.

Similarly, for the function $f_2(\theta)$, it can concluded that the solvability condition (5.1) and the conjugate solution (5.2) have the same form. In this case, it is necessary to set

$$g(\theta) = -[(n_2f_0'^2 + f_0^2)f_0'' + (C_1^2f_0'^2 + C_2^2f_0^2)f_0]/s_0^2.$$

Performing necessary calculations, we establish that the solvability condition is satisfied only for $n_2 = (4s_0 - 1)/(s_0(s_0 - 1)^2)$.

Generalization of the results for an arbitrary coefficient n_k leads to

$$n = 1 + \frac{s_0}{s_0 - 1}\sum_{k=1}^{\infty}\Big(-\frac{\varepsilon}{s_0 - s_*}\Big)^k - \frac{1}{s_0 - 1}\sum_{k=1}^{\infty}\Big(-\frac{\varepsilon}{s_0 - 1}\Big)^k = \frac{s}{s - s_*} - \frac{s}{s - 1},$$

where $s_* = s_0^2/(2s_0 - 1)$.

Solving the obtained equation for s, we find the dependence of the eigenvalue on the material nonlinearity parameter n and the eigenvalue of the linear problem s_0:

$$s = \frac{n(s_0^2 + 2s_0 - 1) + (s_0 - 1)^2}{2n(2s_0 - 1)} + \frac{\sqrt{[n(s_0^2 + 2s_0 - 1) + (s_0 - 1)^2]^2 - 4n^2s_0^2(2s_0 - 1)}}{2n(2s_0 - 1)}.$$

In the case $s_0 = 1/2$, the asymptotic expansion for the material nonlinearity parameter becomes

$$n = 1 - \frac{1}{s_0 - 1}\sum_{k=1}^{\infty}\Big(-\frac{\varepsilon}{s_0 - 1}\Big)^k = -\frac{s}{s - 1},$$

whence we obtain the well-known dependence of the eigenvalue on the nonlinearity parameter that corresponds to the Hutchinson–Rice–Rosengren solution:

$$s = n/(n+1).$$

Conclusions. A method based on perturbation theory was proposed to determine the eigenvalues of the antiplane shear crack problem in a power-law material. It should be noted that the perturbation method for determining the eigenvalues of the crack problem was used in [15], where expressions for the expansion coefficients n_k were derived by eliminating the secular terms in solutions of equations for the function f_k. However, the presence of secular terms in the solution of the problem studied is not a contradiction because the solution is sought on the finite segment $[-\pi, \pi]$ and not on the semi-infinite interval (as is known from perturbation theory, exactly the presence of an infinite region is responsible for the occurrence of nonuniformly suitable expansions — in this case, expansions having secular terms). The second reason for addressing this problem is that the approach developed in [15] cannot be extended to the case of mathematically more complex problems of opening mode and transverse shear cracks. Studies of these mode of loading of cracked solids have shown that the corresponding problems include secular terms of two kinds, whose elimination results in two equations for one unknown quantity n_k in the k-th step. The approach presented in this paper is free from these drawbacks and can be used to solve opening-mode and transverse-shear crack problems.

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